

Uniqueness of the electrostatic solution in Schwarzschild space

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(Received 11 October 2002)

In this Brief Report we give the proof that the solution of any static test charge distribution in Schwarzschild space is unique. In order to give the proof we derive the first Green's identity written with p -forms on (pseudo) Riemannian manifolds. Moreover, the proof of uniqueness can be shown for either any purely electric or purely magnetic field configuration. The spacetime geometry is not crucial for the proof.

PACS numbers: 04.40.Nr, 02.40.Ky, 41.20.Cv

I. INTRODUCTION

In 1971, Cohen and Wald [1] presented the electrostatic field of a point test charge in the Schwarzschild background by using a multipole expansion. For the same problem, Linet found the solution in algebraic form [2]. Recently, Molnár generalized the solution for any test charge distribution in Schwarzschild space with boundary values [3]. He gave the solution not only in algebraic form but also in terms of a multipole expansion. Therefore the question arises whether the solution is unique. In this Brief Report, we present the proof that the solution of any test charge distribution held at rest near a Schwarzschild black hole is unique.

The Brief Report is organized as follows: In Sec. II we derive Green's first identity written with p -forms on (pseudo) Riemannian manifolds. Holmann and Rummeler gave in their book a version for a pair of p and $(p-1)$ -forms on Riemannian manifolds [4]. However, our version is more general because it is valid for a pair of two p -forms and for pseudo-Riemannian manifolds too. The Schwarzschild spacetime, for example, is pseudo-Riemannian. In Sec. III the uniqueness of the electrostatic solution is derived with the help of Green's first identity. In Sec. IV we replace the spherical symmetry by the axial one and show that, for the purely electric or purely magnetic solutions, the proof can be completed along the same lines.

II. THE FIRST GREEN'S IDENTITY

Let (M, g) be an n -dimensional oriented (pseudo) Riemannian manifold and let D be a region of M with smooth boundary such that \bar{D} is compact. For the two p -forms $u, v \in \bigwedge_p(M)$ we consider the combination

$$u \wedge *dv \in \bigwedge_{n-1}(M). \quad (1)$$

In the following we use Stokes' theorem, the anti-Leibniz rule and several facts on p -forms which can be found in the literature, e.g., in [4, 5, 6, 7]. Using Stokes' theorem and the

anti-Leibniz rule we obtain

$$\begin{aligned} \int_{\partial D} u \wedge *dv &= \int_D d(u \wedge *dv) \\ &= \int_D du \wedge *dv + (-1)^p \int_D u \wedge d* dv. \end{aligned} \quad (2)$$

Now, we write the term $d* dv$ in Eq. (2) in a different form. For every form $\omega \in \bigwedge_k(M)$

$$**\omega = (-1)^{k(n-k)} \text{sgn}(g) \omega. \quad (3)$$

This gives us

$$(-1)^{k(k-n)} \text{sgn}(g) **\omega = \omega. \quad (4)$$

With $\omega = d* dv$ and $k = n - p$

$$\begin{aligned} d* dv &= (-1)^{(n-p)(n-p-n)} \text{sgn}(g) **d* dv \\ &= (-1)^{p(p-n)} \text{sgn}(g) **d* dv. \end{aligned} \quad (5)$$

The codifferential $\delta : \bigwedge_q(M) \rightarrow \bigwedge_{q-1}(M)$ is defined by

$$\delta := \text{sgn}(g) (-1)^{nq+n} * d*. \quad (6)$$

We solve Eq. (6) for $*d*$

$$*d* = (-1)^{-nq-n} \text{sgn}(g) \delta. \quad (7)$$

Setting Eq. (7) in Eq. (5) we obtain for $d* dv$ ($q = p + 1$)

$$d* dv = (-1)^p * \delta dv. \quad (8)$$

We set Eq. (8) in Eq. (2)

$$\int_{\partial D} u \wedge *dv = \int_D du \wedge *dv + \int_D u \wedge * \delta dv. \quad (9)$$

Since for two p -forms α, β

$$\alpha \wedge * \beta = \beta \wedge * \alpha,$$

Eq. (9) becomes

$$\int_{\partial D} u \wedge *dv = \int_D du \wedge *dv + \int_D \delta dv \wedge *u. \quad (10)$$

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Next, we consider

$$\delta v \wedge * u \in \bigwedge_{n-1}(M).$$

Using Stokes' theorem and the anti-Leibniz rule again we have

$$\int_{\partial D} \delta v \wedge * u = \int_D d\delta v \wedge * u + (-1)^{p-1} \int_D \delta v \wedge d* u. \quad (11)$$

Now we add Eqs. (11) and (10)

$$\begin{aligned} \int_{\partial D} (u \wedge * dv + \delta v \wedge * u) &= \int_D du \wedge * dv + \int_D \square v \wedge * u \\ &\quad + (-1)^{p-1} \int_D \delta v \wedge d* u. \end{aligned} \quad (12)$$

This is Green's first identity written with p -forms on a (pseudo) Riemannian manifold, where $\square := d \circ \delta + \delta \circ d$ is the Laplace-Beltrami operator.

One can easily derive the second Green's identity [3] with the help of Eq. (12). If we write down Eq. (12) again with u and v interchanged, and subtract it from Eq. (12), we have

$$\begin{aligned} \int_{\partial D} (u \wedge * dv - v \wedge * du + \delta v \wedge * u - \delta u \wedge * v) \\ = \int_D (\square v \wedge * u - \square u \wedge * v) + (-1)^{p-1} \\ \times \int_D (\delta v \wedge d* u - \delta u \wedge d* v). \end{aligned} \quad (13)$$

Note that $du \wedge * dv = dv \wedge * du$. By definition (6) it follows that

$$\begin{aligned} \delta v \wedge d* u &= \text{sgn}(g) (-1)^{np+n} * d* v \wedge d* u \\ &= \text{sgn}(g) (-1)^{np+n} * d* u \wedge d* v = \delta u \wedge d* v. \end{aligned} \quad (14)$$

With the help of Eq. (14) the second term on the right-hand side of Eq. (13) cancels and we find

$$\begin{aligned} \int_{\partial D} (u \wedge * dv - v \wedge * du + \delta v \wedge * u - \delta u \wedge * v) \\ = \int_D (\square v \wedge * u - \square u \wedge * v). \end{aligned} \quad (15)$$

This is Green's second identity which was already derived by Molnár [3].

III. UNIQUENESS OF THE ELECTROSTATIC SOLUTION

We write Maxwell's equations with the exterior calculus

$$dF^{(1)} = 0, \quad \delta F^{(1)} = 4\pi J. \quad (16)$$

By Poincaré's lemma we can introduce a potential form $A^{(1)} = A_\mu^{(1)} dx^\mu$ with $F^{(1)} = dA^{(1)}$. Then, the inhomogeneous Maxwell equations become

$$\delta dA^{(1)} = 4\pi J. \quad (17)$$

The gauge freedom permits us to choose a special gauge condition for $A^{(1)}$. We require the Lorenz condition $\delta A^{(1)} = 0$. Thus, Eq. (17) may be written in the form

$$\square A^{(1)} = 4\pi J. \quad (18)$$

Now, suppose that there exists another solution $A^{(2)}$ satisfying the Lorenz condition, Eq. (18), and the same boundary conditions. Let $A \in \bigwedge_1(M)$

$$A := A^{(2)} - A^{(1)}. \quad (19)$$

Then

$$\square A = 0, \quad \delta A = 0, \quad A|_{\partial D} = 0. \quad (20)$$

We set $A = u = v$ in Green's first identity (12)

$$\begin{aligned} \int_{\partial D} (A \wedge * dA + \delta A \wedge * A) &= \int_D dA \wedge * dA \\ &\quad + \int_D \square A \wedge * A + (-1)^{p-1} \int_D \delta A \wedge d* A. \end{aligned} \quad (21)$$

With the specified properties of A , this reduces to

$$\int_D dA \wedge * dA = 0. \quad (22)$$

In the following we write $F = dA$. $F \in \bigwedge_2(M)$ denotes, like A , the difference of two solutions

$$F \equiv F^{(2)} - F^{(1)}. \quad (23)$$

Thus, Eq. (22) becomes

$$\int_D F \wedge * F = 0. \quad (24)$$

One can show that

$$F \wedge * F = (F, F) \eta, \quad (25)$$

where

$$(F, F) = \frac{1}{2} F_{\mu\nu} g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta} \quad (26)$$

is the scalar product induced in $\bigwedge_2(M)$ and $\eta \in \bigwedge_4(M)$ is the volume element. The uniqueness of solutions of Eq. (18) can only be shown for particular cases where the scalar product is semidefinite.

Now, consider the electrostatic potential $\Phi^{(1)}$ of a static test charge distribution in the Schwarzschild background [2, 3] and suppose that there is another solution $\Phi^{(2)}$. Let Φ be the difference between $\Phi^{(2)}$ and $\Phi^{(1)}$

$$\Phi := \Phi^{(2)} - \Phi^{(1)}. \quad (27)$$

Then, the vector potential is

$$A_\mu = \Phi \delta_\mu^0 \quad (28)$$

and for the Schwarzschild metric we can write

$$g^{\mu\nu} = 0 \quad \text{for} \quad \mu \neq \nu. \quad (29)$$

Thus, Eq. (26) becomes

$$(F, F) = g^{00} g^{ii} (\Phi_{,i})^2 = \text{definite}. \quad (30)$$

Since the scalar product (26) is definite for this special case, we can conclude with Eq. (24) that

$$\Phi_{,i} = 0 \quad \text{for all} \quad i = 1, 2, 3. \quad (31)$$

Consequently, inside D , Φ is constant. For Dirichlet boundary conditions (20), $\Phi = 0$ on ∂D so that, inside D , $\Phi^{(1)} = \Phi^{(2)}$ and the solution is unique.

So far we restricted our analysis to the static case. However, the proof of uniqueness can be straightforwardly generalized to the case of any purely electric or purely magnetic field configuration, because one can infer with the scalar product (26) that all solutions are unique for which F_{0i} or F_{ij} (i. e. E_i or B_k) vanish and the scalar product of \mathbf{E} or \mathbf{B} is definite.

IV. STATIONARY AND AXISYMMETRIC SYSTEMS

We call a system stationary and axisymmetric when all physical quantities, including the metric tensor components $g_{\mu\nu}$, which describe the system are independent of time t and of a toroidal angle φ . We choose a coordinate system (x^μ) with $x^0 = ct$ ($c = 1$), $x^1 = \varphi$, and x^2, x^3 some poloidal coordinates. Assuming in addition that all physical quantities are invariant to the simultaneous inversion of t and φ — which is reasonable for any rotating equilibrium — the most general line element ds can be represented as follows [8]:

$$(ds)^2 = g_{rs} dx^r dx^s + g_{ab} dx^a dx^b, \quad (32)$$

$$g_{ab} = 0 \quad \text{for} \quad a \neq b,$$

where the indices r, s run from 0 to 1, and a, b from 2 to 3. Now, it is useful to choose particular poloidal coordinates, as defined by the poloidal stream lines, $\Psi = \text{const}$, and an angle-like coordinate θ varying along the poloidal stream lines

$$x^2 = \Psi, \quad x^3 = \theta. \quad (33)$$

Then, we denote the projections of the j^μ lines onto the poloidal plane as the lines $\Psi = \text{const}$ and the stream function $\sim I$ of j^a is denoted as a flux function, $I = I(\Psi)$. The continuity equation for j^μ in the poloidal plane is then solved as follows:

$$j^2 = 0, \quad 4\pi\sqrt{-g} j^3 = I'(\Psi), \quad (34)$$

where the prime of I means differentiation with respect to Ψ . Elsässer [9] showed in his paper that Ampère's equation in the poloidal plane becomes

$$\sqrt{-\frac{g_{\text{sym}}}{g_{\text{pol}}}} F_{23} = I(\Psi), \quad (35)$$

where

$$g_{\text{pol}} \equiv \det(g_{ab}), \quad g_{\text{sym}} \equiv \det(g_{rs}).$$

For the difference of two solutions (cf. Eq. (23)) we obtain

$$F_{rs} = \partial_r A_s - \partial_s A_r = 0, \quad F_{ab} = 0, \quad F_{ar} = A_{r,a}. \quad (36)$$

Hence, the scalar product (26) gives us

$$(F, F) = \underbrace{A_{0,a} A_{0,b} g^{ab}}_{<0} \underbrace{g^{00}}_{>0} + \underbrace{A_{1,a} A_{1,b} g^{ab}}_{<0} \underbrace{g^{11}}_{<0} + 2A_{0,a} A_{1,b} g^{ab} g^{01}. \quad (37)$$

We see again that uniqueness is only obtained in general if either A_0 or A_i are zero, i. e. , if either the electrostatic field or the magnetic field vanishes. This is also true for a diagonal metric (Schwarzschild, Minkowski, $g^{01} = 0$). In other words, the symmetry of the spacetime geometry is not crucial for the proof.

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